Experiments
(Equal Probability Randomization)

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Learning goals for today

At the end of this session, you will be able to:

1. Explain exchangeability in randomized experiments
2. Connect causal inference to sampling for population inference

Exchangeable sampling from a population

Exchangeable sampling from a population

Population Outcomes

$Y_{\textbf{Maria}}$
$Y_{\textbf{William}}$
$Y_{\textbf{Rich}}$
$Y_{\textbf{Sarah}}$
$Y_{\textbf{Alondra}}$
$Y_{\textbf{Jesús}}$

Exchangeable sampling from a population

Population Outcomes	Randomized Sampling
$Y_{\textbf{Maria}}$	$S_{\textbf{Maria}} = 1$
$Y_{\textbf{William}}$	$S_{\textbf{William}} = 0$
$Y_{\textbf{Rich}}$	$S_{\textbf{Rich}} = 0$
$Y_{\textbf{Sarah}}$	$S_{\textbf{Sarah}} = 1$
$Y_{\textbf{Alondra}}$	$S_{\textbf{Alondra}} = 0$
$Y_{\textbf{Jesús}}$	$S_{\textbf{Jesús}} = 1$

Exchangeable sampling from a population

Population Outcomes	Randomized Sampling	Sampled Outcomes
Y_{Maria}	$S_{\text{Maria}} = 1$	Y_{Maria}
Y_{William}	$S_{\text{William}} = 0$	
Y_{Rich}	$S_{\text{Rich}} = 0$	
Y_{Sarah}	$S_{\text{Sarah}} = 1$	Y_{Sarah}
Y_{Alondra}	$S_{\text{Alondra}} = 0$	
$Y_{\text{Jesús}}$	$S_{\text{Jesús}} = 1$	$Y_{\text{Jesús}}$

Exchangeable sampling from a population

Population Outcomes	Randomized Sampling	Sampled Outcomes
Y_{Maria}	$S_{\text{Maria}} = 1$	Y_{Maria}
Y_{William}	$S_{\text{William}} = 0$	
Y_{Rich}	$S_{\text{Rich}} = 0$	
Y_{Sarah}	$S_{\text{Sarah}} = 1$	Y_{Sarah}
Y_{Alondra}	$S_{\text{Alondra}} = 0$	
$Y_{\text{Jesús}}$	$S_{\text{Jesús}} = 1$	$Y_{\text{Jesús}}$

Estimator:
Estimate the population mean by the sample mean

Key assumption:
Sampled and unsampled units are **exchangeable** due to random sampling

$$Y \perp\!\!\!\perp S$$

Now suppose our population all participate
in a randomized experiment with
treatment ($A = 1$) and control ($A = 0$) conditions

Exchangeable treatment assignment

Population Potential Outcomes

Y^1_{Maria}
Y^1_{William}
Y^1_{Rich}
Y^1_{Sarah}
Y^1_{Alondra}
$Y^1_{\text{Jesús}}$

Exchangeable treatment assignment

Population Potential Outcomes	**Randomized Treatment**
Y^1_{Maria}	$A_{\text{Maria}} = 1$
Y^1_{William}	$A_{\text{William}} = 0$
Y^1_{Rich}	$A_{\text{Rich}} = 0$
Y^1_{Sarah}	$A_{\text{Sarah}} = 1$
Y^1_{Alondra}	$A_{\text{Alondra}} = 0$
$Y^1_{\text{Jesús}}$	$A_{\text{Jesús}} = 1$

Exchangeable treatment assignment

Population Potential Outcomes	Randomized Treatment	Observed Outcomes
Y^1_{Maria}	$A_{\text{Maria}} = 1$	Y^1_{Maria}
Y^1_{William}	$A_{\text{William}} = 0$	
Y^1_{Rich}	$A_{\text{Rich}} = 0$	
Y^1_{Sarah}	$A_{\text{Sarah}} = 1$	Y^1_{Sarah}
Y^1_{Alondra}	$A_{\text{Alondra}} = 0$	
$Y^1_{\text{Jesús}}$	$A_{\text{Jesús}} = 1$	$Y^1_{\text{Jesús}}$

Exchangeable treatment assignment

Population Potential Outcomes	Randomized Treatment	Observed Outcomes
Y^1_{Maria}	$A_{\text{Maria}} = 1$	Y^1_{Maria}
Y^1_{William}	$A_{\text{William}} = 0$	
Y^1_{Rich}	$A_{\text{Rich}} = 0$	
Y^1_{Sarah}	$A_{\text{Sarah}} = 1$	Y^1_{Sarah}
Y^1_{Alondra}	$A_{\text{Alondra}} = 0$	
$Y^1_{\text{Jesús}}$	$A_{\text{Jesús}} = 1$	$Y^1_{\text{Jesús}}$

Estimator:
Estimate the population mean $\text{E}(Y^1)$ by the untreated sample mean

Key assumption:
Treated and untreated units are **exchangeable** due to random treatment assignment

$$Y^1 \perp\!\!\!\perp A$$

Exchangeable treatment assignment

Population Potential Outcomes	Randomized Treatment	Observed Outcomes
Y^0_{Maria}	$A_{\text{Maria}} = 1$	
Y^0_{William}	$A_{\text{William}} = 0$	Y^0_{William}
Y^0_{Rich}	$A_{\text{Rich}} = 0$	Y^0_{Rich}
Y^0_{Sarah}	$A_{\text{Sarah}} = 1$	
Y^0_{Alondra}	$A_{\text{Alondra}} = 0$	Y^0_{Alondra}
$Y^0_{\text{Jesús}}$	$A_{\text{Jesús}} = 1$	

Estimator:
Estimate the population mean $\text{E}(Y^0)$ by the untreated sample mean

Key assumption:
Treated and untreated units are **exchangeable** due to random treatment assignment

$Y^0 \perp\!\!\!\perp A$

Exchangeable treatment assignment

Population Potential Outcomes		Randomized Treatment	Observed Outcomes	
Y^1_{Maria}	Y^0_{Maria}	$A_{\text{Maria}} = 1$	Y^1_{Maria}	
Y^1_{William}	Y^0_{William}	$A_{\text{William}} = 0$		Y^0_{William}
Y^1_{Rich}	Y^0_{Rich}	$A_{\text{Rich}} = 0$		Y^0_{Rich}
Y^1_{Sarah}	Y^0_{Sarah}	$A_{\text{Sarah}} = 1$	Y^1_{Sarah}	
Y^1_{Alondra}	Y^0_{Alondra}	$A_{\text{Alondra}} = 0$		Y^0_{Alondra}
$Y^1_{\text{Jesús}}$	$Y^0_{\text{Jesús}}$	$A_{\text{Jesús}} = 1$	$Y^1_{\text{Jesús}}$	

Exchangeable treatment assignment

Causal Estimand:

(expected outcome if assigned to treatment)
− (expected outcome if assigned to control)

$$\mathrm{E}\left(Y^1\right) - \mathrm{E}\left(Y^0\right)$$

Exchangeability Assumption:

Potential outcomes are independent of treatment assignment

$$\{Y^0, Y^1\} \perp\!\!\!\perp A$$

Empirical Estimand:

(expected outcome among the treated)
− (expected outcome among the untreated)

$$\mathrm{E}(Y \mid A = 1) - \mathrm{E}(Y \mid A = 0)$$

Exchangeable treatment assignment: Proof

$$
\begin{aligned}
&\mathrm{E}\left(Y^1\right) - \mathrm{E}\left(Y^0\right) \\
&= \mathrm{E}\left(Y^1 \mid A = 1\right) - \mathrm{E}\left(Y^0 \mid A = 0\right) \\
&= \mathrm{E}\left(Y \mid A = 1\right) - \mathrm{E}\left(Y \mid A = 0\right)
\end{aligned}
$$

Exchangeable treatment assignment: Proof

$$\begin{aligned} & \mathrm{E}\left(Y^1\right) - \mathrm{E}\left(Y^0\right) \\ & = \mathrm{E}\left(Y^1 \mid A = 1\right) - \mathrm{E}\left(Y^0 \mid A = 0\right) \\ & = \mathrm{E}\left(Y \mid A = 1\right) - \mathrm{E}\left(Y \mid A = 0\right) \qquad \text{by consistency} \end{aligned}$$

Exchangeable treatment assignment: Proof

$$
\begin{aligned}
&\mathrm{E}\left(Y^1\right) - \mathrm{E}\left(Y^0\right) \\
&= \mathrm{E}\left(Y^1 \mid A = 1\right) - \mathrm{E}\left(Y^0 \mid A = 0\right) && \text{by exchangeability} \\
&= \mathrm{E}\left(Y \mid A = 1\right) - \mathrm{E}\left(Y \mid A = 0\right) && \text{by consistency}
\end{aligned}
$$

Exchangeable treatment assignment: Proof

$$
\begin{aligned}
&\mathrm{E}\left(Y^1\right) - \mathrm{E}\left(Y^0\right) \\
&= \mathrm{E}\left(Y^1 \mid A = 1\right) - \mathrm{E}\left(Y^0 \mid A = 0\right) \quad \text{by exchangeability} \\
&= \mathrm{E}\left(Y \mid A = 1\right) - \mathrm{E}\left(Y \mid A = 0\right) \quad \text{by consistency}
\end{aligned}
$$

This is an example of **causal identification**:
using assumptions to arrive at an empirical quantity
(involving only factual random variables, no potential outcomes)
that equals our causal estimand if the assumptions hold

The causal estimand $\mathrm{E}(Y^1) - \mathrm{E}(Y^0)$ is **identified** by the empirical estimand $\mathrm{E}(Y \mid A = 1) - \mathrm{E}(Y \mid A = 0)$