Longitudinal Weighting and Marginal Structural Models

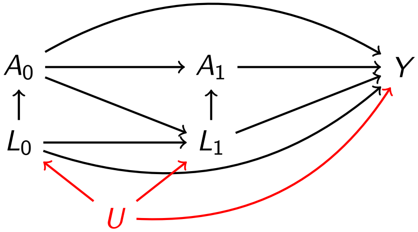
Ian Lundberg

Learning goals for today

At the end of class, you will be able to:

1. Reason about the sequential ignorability assumption
2. Apply inverse probability weighting to treatments over time

Identification: The adjustment set



A joint adjustment set for $\bar{A}$ is doomed

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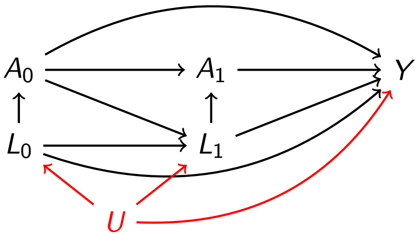

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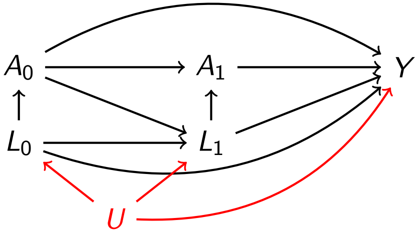

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- What happens if you adjust for L_1?
  - You block a causal path: $A_0 \rightarrow \boxed{L_1} \rightarrow Y$
  - You open a backdoor path: $A_0 \rightarrow \boxed{L_1} \leftarrow U \rightarrow Y$

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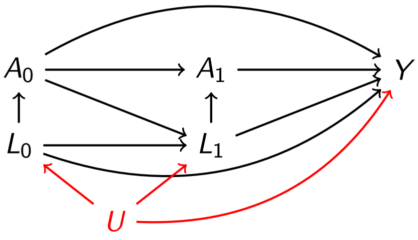

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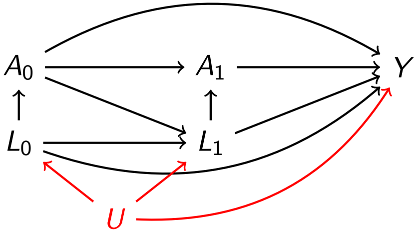

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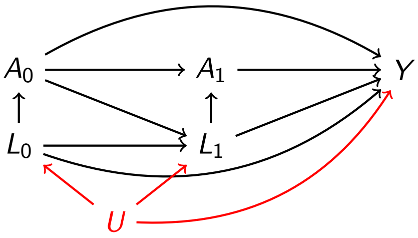

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What to do?

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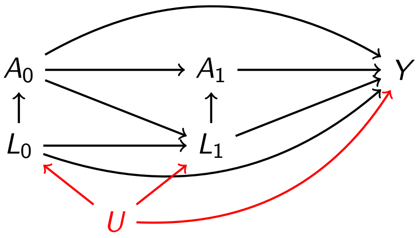

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What to do? [Class Exercise]

Generalizing the class exercise

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An ideal case: The **sequentially randomized experiment**

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1. randomize treatment at time 0, then

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Then you can estimate $\text{E}(Y^{a_1,\dots,a_k})$ by $\text{E}(Y \mid \vec{A} = \vec{a})$.

Generalizing the class exercise: Conditional assignments

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A **sequential conditionally randomized experiment**

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Generalizing the class exercise: Conditional assignments

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3. ...
4. Repeat up to time k

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Then you can estimate $\text{E}(Y^{a_1,\dots,a_k})$ by the methods to come

Notation

- $\bar{A}_k = (A_0, A_1, \ldots, A_k)$ treatments up to time k
- $\bar{L}_k = (L_0, L_1, \ldots, L_k)$ confounders up to time k
- $g()$ treatment strategy
- Y^g potential outcome under that strategy

Identification: Sequential ignorability

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Potential outcome under assignment rule g	is independent of	treatment at time k	given	treatments up to $k-1$ followed rule g	and confounders up to time k
Y^g	$\perp\!\!\!\perp$	A_k	$\mid$	$\bar{A}_{k-1} = g(\bar{A}_{k-2}, \bar{L}_{k-1}),$	$\bar{L}_k$

$$Y^g \perp\!\!\!\perp A_k \mid \bar{A}_{k-1} = g(\bar{A}_{k-2}, \bar{L}_{k-1}), \bar{L}_k$$

for all assignment rules g and time periods $k = 1, \ldots, K$

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$$
\begin{tikzcd}
A_0 \arrow[r] \arrow[dr] \arrow[rr, bend left] & A_1 \arrow[r] & Y \\
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\end{tikzcd}
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$$\overset{\text{Potential outcome under assignment rule } g}{Y^g} \quad \overset{\text{is independent of}}{\perp\!\!\!\perp} \quad \overset{\text{treatment at time } k}{A_k} \quad \overset{\text{given}}{\Big|} \quad \overset{\text{treatments up to } k-1 \text{ followed rule } g}{\bar{A}_{k-1} = g(\bar{A}_{k-2}, \bar{L}_{k-1}),} \quad \overset{\text{and confounders up to time } k}{\bar{L}_k}$$

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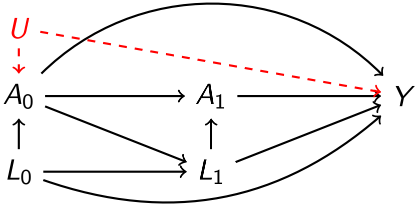

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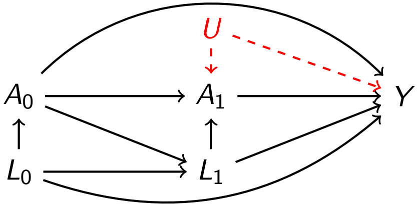

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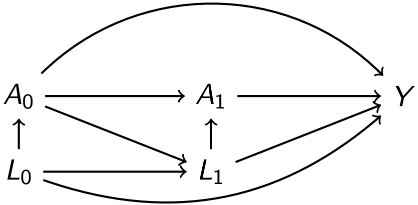

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\end{tikzcd}$$

Holds by design in sequentially randomized experiments.
Holds by assumption in observational studies.

Estimation: Two strategies

1. Inverse probability weighting (+ marginal structural models)
2. Structural nested mean models (coming next class)

Inverse probability weighting: DAG motivation

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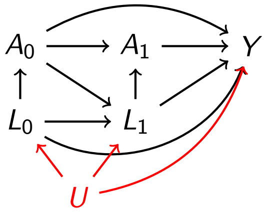
We observe data from this model



Inverse probability weighting: DAG motivation

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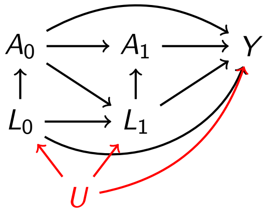
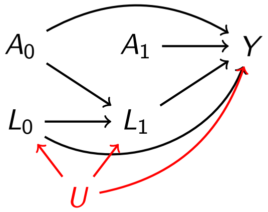


We want this



Inverse probability weighting

In time 0, define an inverse probability of treatment weight such that $A_0 \perp\!\!\!\perp L_0$ in the weighted pseudo-population

$$W^{A_0} = \frac{1}{\mathrm{P}(A_0 \mid L_0)}$$

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Repeat in every time period t

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$$W^{A_t} = \frac{1}{\mathrm{P}(A_t \mid \bar{A}_{t-1}, \bar{L}_t)}$$

Define the overall weight as the product

$$W^{\bar{A}} = \prod_{k=0}^{K} \frac{1}{\mathrm{P}(A_k \mid \bar{A}_{k-1}, \bar{L}_k)}$$

Inverse probability weighting

What did we accomplish?

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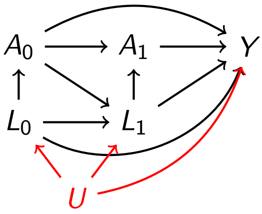

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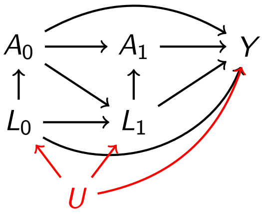

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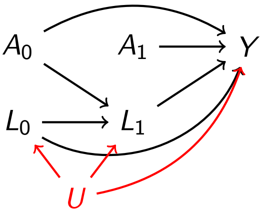

Real example: Neighborhood disadvantage

Wodtke et al. 2011

Wodtke, G. T., Harding, D. J., & Elwert, F. (2011).
Neighborhood effects in temporal perspective: The impact of long-term exposure to concentrated disadvantage on high school graduation.
American Sociological Review, 76(5), 713-736.

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How does the neighborhood in which a child lives affect that child's probability of high school completion?

- Define a neighborhood as a Census tract
- Score that neighborhood along several dimensions
  - poverty
  - unemployment
  - welfare receipt
  - female-headed households
  - education
  - occupational structure

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This 5-value treatment is "neighborhood disadvantage"

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$$\bar{a}$$

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The authors study the effect of neighborhood disadvantage,

$$\begin{aligned} E(Y_{\bar{a}} - Y_{\bar{a}'}) &= E(Y_{\bar{a}}) - E(Y_{\bar{a}'}) \qquad (1) \\ &= P(Y_{\bar{a}} = 1) - P(Y_{\bar{a}'} = 1), \end{aligned}$$

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Example:
$\bar{a}$ is residence in the most advantaged neighborhood each year
and
$\bar{a}'$ is residence in the most disadvantaged neighborhood each year

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Problem: Neighborhoods A_1 shape family characteristics L_2, which confound where people live in the future A_2

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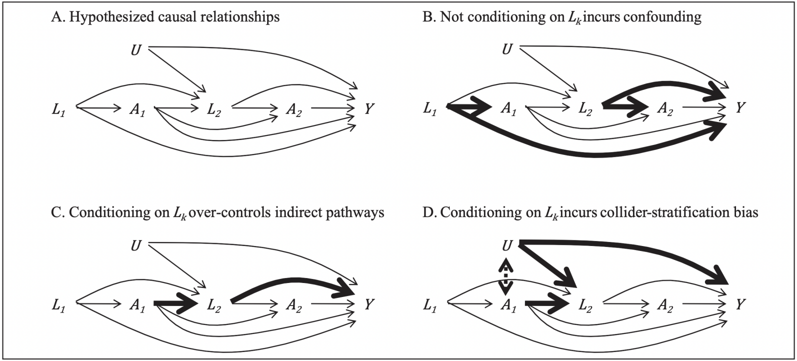

Figure 1. Causal Graphs for Exposure to Disadvantaged Neighborhoods with Two Waves of Follow-up

Note: A_k = neighborhood context, L_k = observed time-varying confounders, U = unobserved factors, Y = outcome.

Table 2. Time-Dependent Sample Characteristics

Variable	Blacks (n = 834)			Nonblacks (n = 1,259)		
	Age 1	Age 10	Age 17	Age 1	Age 10	Age 17
NH disadvantage index, percent						
1st quintile	3.48	3.60	3.48	13.34	19.14	20.65
2nd quintile	3.24	3.72	6.00	19.46	18.67	21.84
3rd quintile	5.28	5.88	7.79	26.13	23.27	22.48
4th quintile	14.87	18.11	18.47	26.13	23.99	21.13
5th quintile	73.14	68.71	64.27	14.93	14.93	13.90
FU head's marital status, percent						
Unmarried	33.93	44.84	52.04	5.88	11.36	15.09
Married	66.07	55.16	47.96	94.12	88.64	84.91
FU head's employment status, percent						
Unemployed	27.22	32.61	33.09	8.10	8.02	9.69
Employed	72.78	67.39	66.91	91.90	91.98	90.31
Public assistance receipt, percent						
Did not receive AFDC	81.06	75.66	82.37	96.27	96.19	97.93
Received AFDC	18.94	24.34	17.63	3.73	3.81	2.07
Homeownership, percent						
Do not own home	69.66	53.48	50.12	40.19	22.32	20.73
Own home	30.34	46.52	49.88	59.81	77.68	79.27
FU income in $1,000s, mean	19.68	25.04	27.45	32.59	46.65	57.50
FU head's work hours, mean	30.08	26.82	27.51	42.65	40.84	40.68
FU size, mean	5.75	5.32	4.81	4.22	4.69	4.33
Cum. residential moves, mean	.32	2.48	3.64	.32	2.16	3.02

Note: NH = neighborhood; FU = family unit. Statistics reported for children not lost to follow-up before age 20 (first imputation dataset).

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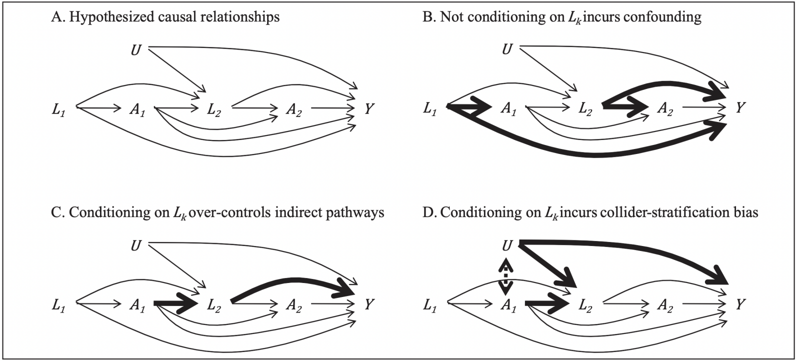

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Wodtke et al. 2011

Solution: MSM-IPW

$$w_i = \prod_{k=1}^{K} \frac{1}{P(A_k = a_{ki} \mid \bar{A}_{k-1} = \bar{a}_{(k-1)i}, \bar{L}_k = \bar{l}_{ki})}. \quad (4)$$

Real example: Neighborhood disadvantage

Wodtke et al. 2011

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Also with stabilized weights

$$sw_i = \prod_{k=1}^{K} \frac{P(A_k = a_{ki} \mid \bar{A}_{k-1} = \bar{a}_{(k-1)i}, L_0 = l_0)}{P(A_k = a_{ki} \mid \bar{A}_{k-1} = \bar{a}_{(k-1)i}, \bar{L}_k = \bar{l}_{ki})}, \quad (5)$$

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Problem: Huge number of treatments

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- 5 levels of neighborhood disadvantage

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Wodtke et al. 2011

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- $5^{16} = 152,587,890,625$ possible treatment vectors $\vec{A}$
  - For reference: Only 117 billion people have ever been born on Earth

Digression: Marginal structural models in dynamic settings

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Recall that when A takes many values, we can fit a marginal structural model

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- Estimate by $\mathsf{E}^{\mathsf{PP}}(Y \mid A = a)$ where E^{PP} is the expectation in the pseudopopulation weighted so that treatment is independent of confounders.

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Wodtke et al. 2011

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Interpretation: $\bar{a}$ is duration-weighted exposure

Results: Neighborhood disadvantage

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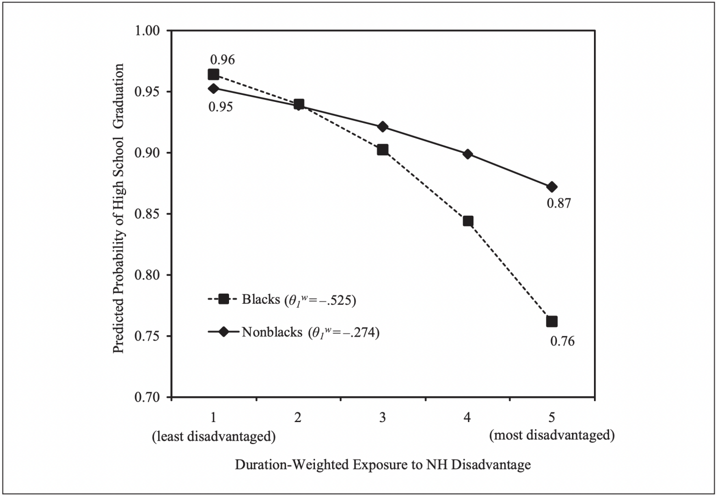

Figure 3. Predicted Probability of High School Graduation by Neighborhood Exposure History

Note: NH = Neighborhood

Learning goals for today

At the end of class, you will be able to:

1. Reason about the sequential ignorability assumption
2. Apply inverse probability weighting to treatments over time