Estimation by weighting

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Learning goals for today

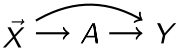
At the end of class, you will be able to estimate average causal effects by modeling treatment assignment probabilities.

Optional reading:

- Hernán and Robins 2020 Chapter 12.1–12.5, 13, 15.1

Review of what we have learned

Causal assumptions



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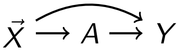

Nonparametric estimator

- Group by L, then mean difference in Y over A
- Re-aggregate over subgroups

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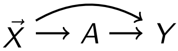

Nonparametric estimator

- Group by L, then mean difference in Y over A
- Re-aggregate over subgroups

Outcome modeling estimator

- Model Y^1 given L among the treated
- Model Y^0 given L among the untreated
- Predict for everyone and take the difference
- Average over all units

Inverse probability weighting: Population mean

Population Outcomes	Randomized Sampling	Sampled Outcomes
$Y_{\textbf{Maria}}$	$S_{\textbf{Maria}} = 1$	$Y_{\textbf{Maria}}$
$Y_{\textbf{William}}$	$S_{\textbf{William}} = 0$	
$Y_{\textbf{Rich}}$	$S_{\textbf{Rich}} = 0$	
$Y_{\textbf{Sarah}}$	$S_{\textbf{Sarah}} = 1$	$Y_{\textbf{Sarah}}$
$Y_{\textbf{Alondra}}$	$S_{\textbf{Alondra}} = 0$	
$Y_{\textbf{Jesús}}$	$S_{\textbf{Jesús}} = 1$	$Y_{\textbf{Jesús}}$

How many people do Maria, Sarah, and Jesús each represent?

Inverse probability weighting: Population mean

	Population Outcomes	Randomized Sampling	Sampled Outcomes
No Parent Completed College	Y_{Maria}	$S_{\text{Maria}} = 1$	Y_{Maria}
	Y_{William}	$S_{\text{William}} = 0$	
	Y_{Rich}	$S_{\text{Rich}} = 0$	
A Parent Completed College	Y_{Sarah}	$S_{\text{Sarah}} = 1$	Y_{Sarah}
	Y_{Alondra}	$S_{\text{Alondra}} = 0$	
	$Y_{\text{Jesús}}$	$S_{\text{Jesús}} = 1$	$Y_{\text{Jesús}}$

How many people do Maria, Sarah, and Jesús each represent?

Inverse probability weighting: Population mean

Each unit has a probability of being sampled.

$$P(S = 1 \mid \vec{X})$$

If we believe conditionally exchangeable sampling,

$$S \perp\!\!\!\perp Y \mid \vec{X}$$

weight by the inverse probability of sampling.

$$w = \frac{1}{\text{P}(S = 1 \mid \vec{X})}$$

$$\hat{\text{E}}(Y) = \frac{\sum_i w_i y_i}{\sum_i w_i}$$

Inverse probability weighting: Non-probability sample

Suppose we have the Xbox sample (Wang et al. 2015)

- Imagine we believe conditional exchangeability
- They have the counts $n_{\vec{x}}$ in each demographic subgroup $\vec{x}$ in the sample
- They estimate the population sizes $N_{\vec{x}}$ from exit polls
- Can we estimate by weighting?
  - Assume for simplicity that each $n_{\vec{x}}$ is much greater than 0

Inverse probability weighting: Non-probability sample

1. Estimate the probability of sampling

$$\hat{\pi}_i = \hat{\text{P}}(S = 1 \mid \vec{X} = \vec{x}_i) = \frac{n_{\vec{X}=\vec{x}_i}}{N_{\vec{X}=\vec{x}_i}} = \frac{\overbrace{\sum_j S_j \mathbb{I}(\vec{X}_j = \vec{x}_i)}^{\text{Number of sample members who look like unit } i}}{\underbrace{\sum_j \mathbb{I}(\vec{X}_j = \vec{x}_i)}_{\text{Number of population members who look like unit } i}}$$

2. Weight by inverse probability of sampling

$$\hat{\text{E}}(Y) = \frac{\sum_i \hat{w}_i y_i}{\sum_i \hat{w}_i} \qquad \text{for } \hat{w}_i = \frac{1}{\hat{\pi}_i}$$

Inverse probability weighting: Non-probability sample

Takeaway: Exactly like a probability sample except

- conditional exchangeability holds only by assumption
- inverse probability of sampling weights must be estimated

Inverse probability weighting: Mean under treatment

$A = 1$ indicates child completed college

Population Outcomes	Randomized Sampling	Sampled Treatment
Y^1_{Maria}	$A_{\text{Maria}} = 1$	Y^1_{Maria}
Y^1_{William}	$A_{\text{William}} = 0$	
Y^1_{Rich}	$A_{\text{Rich}} = 0$	
Y^1_{Sarah}	$A_{\text{Sarah}} = 1$	Y^1_{Sarah}
Y^1_{Alondra}	$A_{\text{Alondra}} = 0$	
$Y^1_{\text{Jesús}}$	$A_{\text{Jesús}} = 1$	$Y^1_{\text{Jesús}}$

How many people do Maria, Sarah, and Jesús each represent?

Inverse probability weighting: Mean under treatment

$A = 1$ indicates child completed college

	Population Outcomes	Randomized Treatment	Sampled Outcomes
No Parent Completed College	Y^1_{Maria}	$A_{\text{Maria}} = 1$	Y^1_{Maria}
	Y^1_{William}	$A_{\text{William}} = 0$	
	Y^1_{Rich}	$A_{\text{Rich}} = 0$	
A Parent Completed College	Y^1_{Sarah}	$A_{\text{Sarah}} = 1$	Y^1_{Sarah}
	Y^1_{Alondra}	$A_{\text{Alondra}} = 0$	
	$Y^1_{\text{Jesús}}$	$A_{\text{Jesús}} = 1$	$Y^1_{\text{Jesús}}$

How many people do Maria, Sarah, and Jesús each represent?

Inverse probability weighting: Mean under treatment

$A = 1$ indicates child completed college. $\vec{X}$ indicates parent completed college.

When estimating the mean outcome under treatment,

$$\text{E}(Y^1)$$

each unit has a probability of being treated.

$$P(A = 1 \mid \vec{X})$$

Weight treated units by the inverse probability of treatment.

$$w = \frac{A}{\text{P}(A = 1 \mid \vec{X})}$$

Inverse probability weighting: Mean under control

$A = 1$ indicates child completed college

	Population Outcomes	Randomized Treatment	Sampled Outcomes
No Parent Completed College	Y^0_{Maria}	$A_{\text{Maria}} = 1$	
	Y^0_{William}	$A_{\text{William}} = 0$	Y^0_{William}
	Y^0_{Rich}	$A_{\text{Rich}} = 0$	Y^0_{Rich}
A Parent Completed College	Y^0_{Sarah}	$A_{\text{Sarah}} = 1$	
	Y^0_{Alondra}	$A_{\text{Alondra}} = 0$	Y^0_{Alondra}
	$Y^0_{\text{Jesús}}$	$A_{\text{Jesús}} = 1$	

How many people do William, Rich, and Alondra each represent?

Inverse probability weighting: Mean under control

$A = 1$ indicates child completed college. $\vec{X}$ indicates parent completed college.

When estimating the mean outcome under treatment,

$$\mathrm{E}(Y^0)$$

each unit has a probability of being untreated.

$$P(A = 0 \mid \vec{X})$$

Weight treated units by the inverse probability of treatment.

$$w = \frac{1 - A}{\mathrm{P}(A = 0 \mid \vec{X})}$$

Inverse probability weighting: Average causal effect

Define inverse probability of treatment weights

$$w_i = \begin{cases} \frac{1}{\mathrm{P}(A=1|\vec{X}=\vec{x}_i)} & \text{if treated} \\ \frac{1}{\mathrm{P}(A=0|\vec{X}=\vec{x}_i)} & \text{if untreated} \end{cases}$$

Estimate each mean potential outcome by a weighted mean

$$\hat{\mathrm{E}}(Y^1) = \sum_{i:A_i=1} w_i Y_i \quad / \quad \sum_{i:A_i=1} w_i$$

$$\hat{\mathrm{E}}(Y^0) = \sum_{i:A_i=0} w_i Y_i \quad / \quad \sum_{i:A_i=0} w_i$$

Take the difference between $\hat{\mathrm{E}}(Y^1)$ and $\hat{\mathrm{E}}(Y^0)$

Exercise: Weight for ATT

Goal: Average treatment effect on the treated

When $X = 1$,

- 7 treated units
- 3 untreated units
- $\text{P}(A = 1 \mid X = 1) = 0.7$

When $X = 0$,

- 4 treated units
- 6 untreated units
- $\text{P}(A = 1 \mid X = 0) = 0.4$

Each treated unit weighted by 1. Total untreated weight at each x should equal total treated weight.

Inverse probability weighting: Experiment

Takeaway:

- weight = inverse probability of observed treatment condition
- estimate by weighted means

Inverse probability weighting: Observational study

Now treatment is not randomly assigned. How do we use weighting?

Inverse probability weighting: Observational study

Now treatment is not randomly assigned. How do we use weighting?

- assume conditionally exchangeable treatment assignment
- estimate inverse probability of treatment weights

Inverse probability weighting: Observational study

Model probability of treatment

$$\hat{\mathsf{P}}(A = 1 \mid \vec{X}) = \text{logit}^{-1}\left(\hat{\alpha} + \hat{\vec{\gamma}}\vec{X}\right)$$

Estimate inverse probability of treatment weights

$$\hat{w}_i = \begin{cases} \frac{1}{\hat{\mathsf{P}}(A=1 \mid \vec{X}=\vec{x}_i)} & \text{if treated} \\ \frac{1}{\hat{\mathsf{P}}(A=0 \mid \vec{X}=\vec{x}_i)} & \text{if untreated} \end{cases}$$

Estimate each mean potential outcome by a weighted mean

$$\hat{\mathsf{E}}(Y^1) = \sum_{i:A_i=1} \hat{w}_i Y_i \quad / \quad \sum_{i:A_i=1} w_i$$

$$\hat{\mathsf{E}}(Y^0) = \sum_{i:A_i=0} \hat{w}_i Y_i \quad / \quad \sum_{i:A_i=0} w_i$$

Unequal sampling and unequal treatment assignment

Unequal sampling and unequal treatment assignment

Unit i was sampled with probability 0.25.

$$\mathrm{P}(S = 1 \mid \vec{X} = \vec{x}_i) = \frac{1}{4} = 0.25$$
$$w_i^{\text{Sampling}} = 4$$

Unequal sampling and unequal treatment assignment

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$$w_i^{\text{Sampling}} = 4$$

Given sampling, received treatment with probability 0.33.

$$\text{P}(A = 1 \mid \vec{X} = \vec{x}_i, S = 1) = \frac{1}{3} = 0.33$$
$$w_i^{\text{Treatment}} = 3$$

Unequal sampling and unequal treatment assignment

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How many population Y^1 values does unit i represent?

Unequal sampling and unequal treatment assignment

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$$w_i^{\text{Treatment}} = 3$$

How many population Y^1 values does unit i represent?

$$w_i^{\text{Sampling}} \times w_i^{\text{Treatment}} = 4 \times 3 = 12$$

Unequal sampling and unequal treatment assignment

In math: To observe Y^1, a unit must be sampled and treated.

$$\begin{aligned}
\text{P}(\text{Observe } Y^1 \mid \vec{X}) &= \text{P}(S = 1, A = 1 \mid \vec{X}) \\
&= \text{P}(A = 1 \mid S = 1, \vec{X})\text{P}(S = 1 \mid \vec{X})
\end{aligned}$$

Unequal sampling and unequal treatment assignment

In math: To observe Y^1, a unit must be sampled and treated.

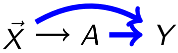
$$\begin{aligned} \text{P}(\text{Observe } Y^1 \mid \vec{X}) &= \text{P}(S = 1, A = 1 \mid \vec{X}) \\ &= \text{P}(A = 1 \mid S = 1, \vec{X})\text{P}(S = 1 \mid \vec{X}) \end{aligned}$$

The inverse probability weight is thus the product of sampling and treatment weights.

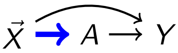
$$\frac{1}{\text{P}(\text{Observe } Y^1 \mid \vec{X})} = \underbrace{\frac{1}{\text{P}(A = 1 \mid S = 1, \vec{X})}}_{\substack{\text{inverse probability} \\ \text{of treatment weight}}} \times \underbrace{\frac{1}{\text{P}(A = 1 \mid S = 1, \vec{X})}}_{\substack{\text{inverse probability} \\ \text{of sampling weight}}}$$

Outcome and treatment modeling: A visual summary

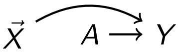
Outcome modeling: Model Y^0 and Y^1 given $\vec{X}$



Treatment modeling: Model A given $\vec{X}$. Reweight.



Original population



Reweighted population

What are the advantages of each strategy? How to choose?

1. Outcome modeling
    - Model Y^1 and Y^0 given $\vec{X}$
    - Predict for everyone
    - Unweighted average
2. Treatment modeling
    - Model A given X
    - Create weights: how many units each case represents
    - Weighted average

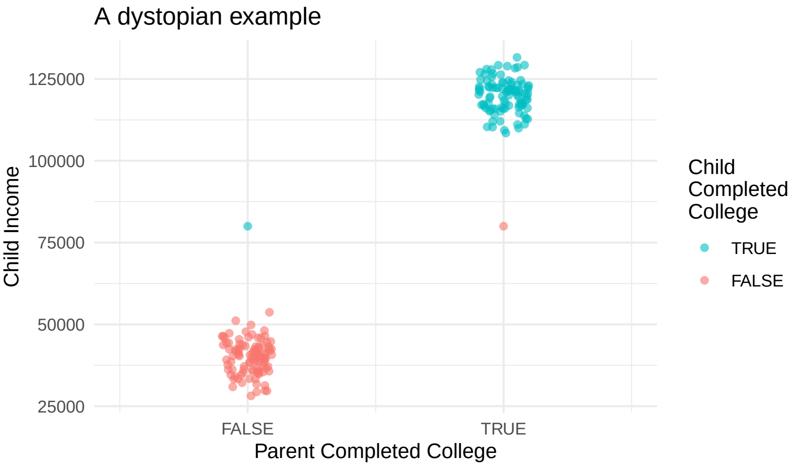
An advantage of treatment modeling

how most social scientists think about research:
model the outcome

Advantages of each strategy: Treatment modeling

- how we already think about population sampling: reweight observed cases to learn about all cases
- transparency about influential observations

Transparency about influential observations



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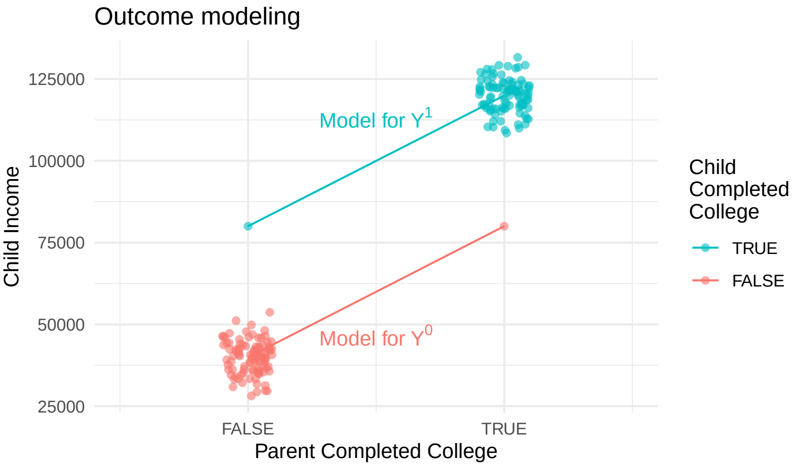

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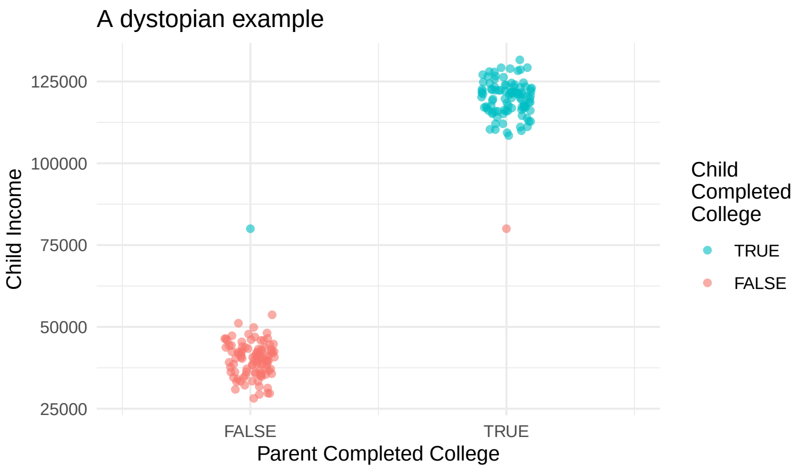

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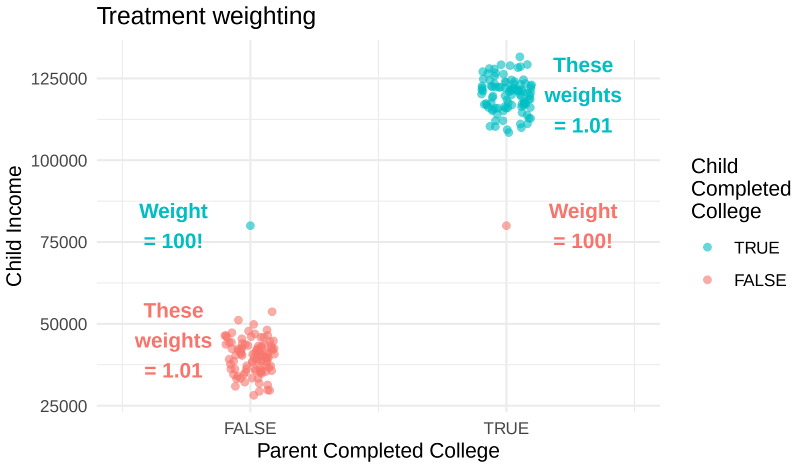
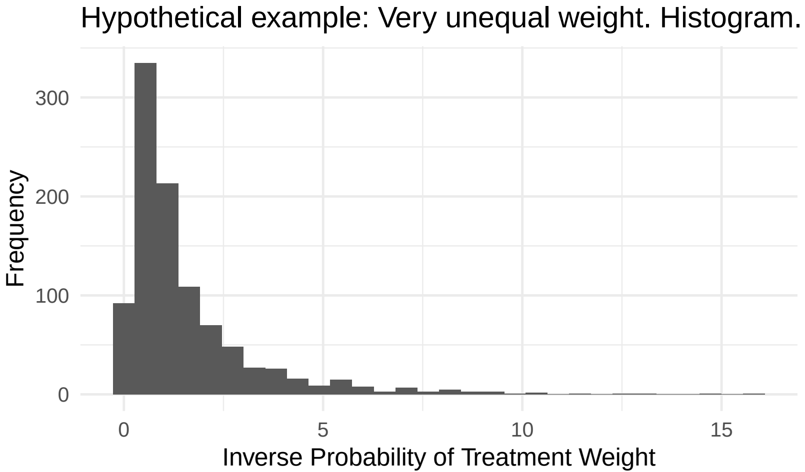


What to do when some weights are big?



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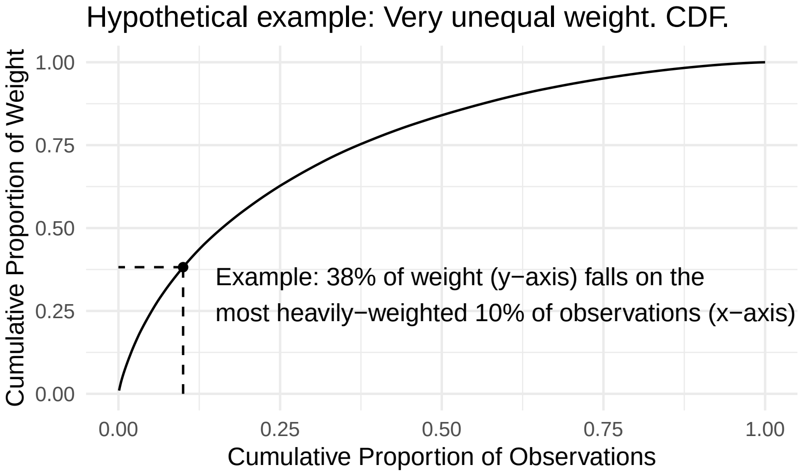

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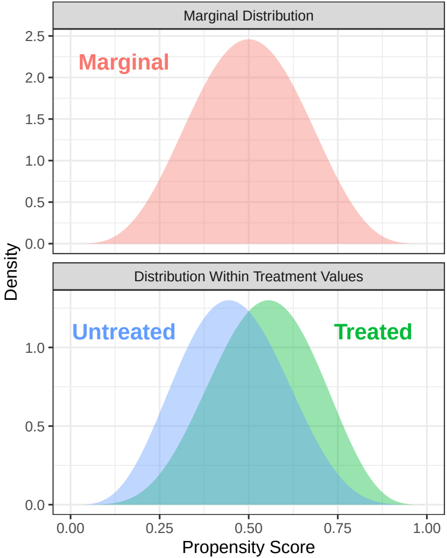
Focus on a feasible subpopulation: Region of common support



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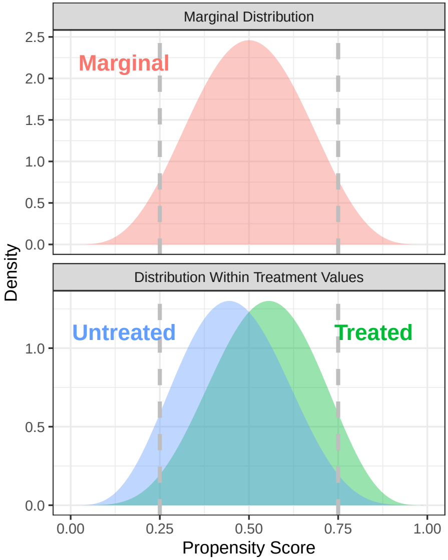

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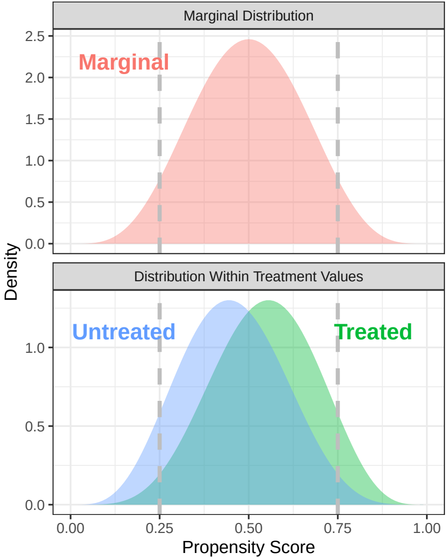

Restrict to a subgroup

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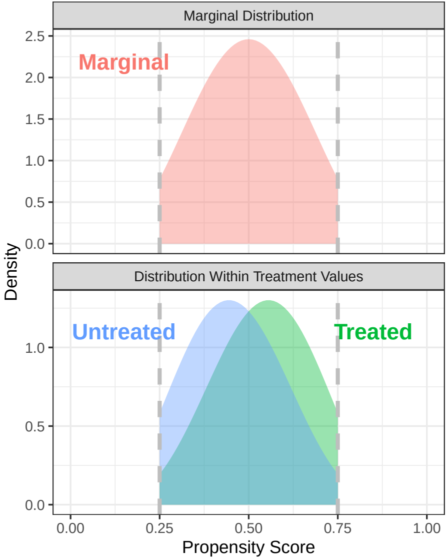

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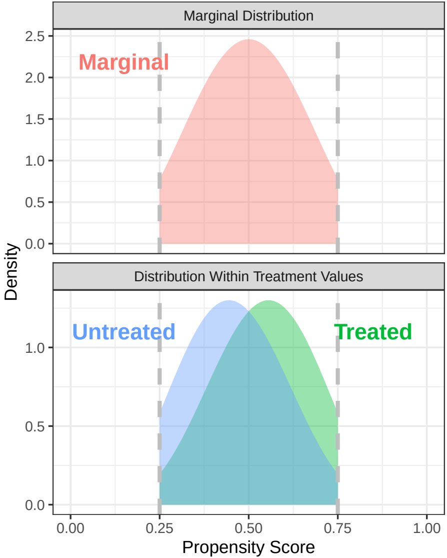

Restrict to a subgroup

Estimate in the subgroup

$$E\left(Y^1 - Y^0 \mid k_1 < P(A = 1 \mid \vec{X}) < k_2\right)$$

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